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GUARANTEED RESULT IN A DIFFERENTIAL GAME WITH A TERMINAL PAYOFF FUNCTION†

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A method for solving game-type control problems with a terminal payoff function is proposed. It consists of applying the ideas of Fenchel–Moreau duality [1] to the general scheme of the method of resolvent functions [2]. The main point of the method is that the resolvent function can be expressed in terms of the conjugate of the payoff function; then, using the involutive property of the conjugation operator for a convex closed function, one obtains a guaranteed estimate for the terminal value of the payoff function, expressed in terms of the initial value of the payoff and the integral of the resolvent function.

This paper develops ideas presented in [2–4], touches on the topics dealt with in [5–9] and suggests new possibilities for the application of convex analysis to the solution of game-type control problems.

1. FORMULATION OF THE PROBLEM AND AUXILIARY RESULTS

Suppose we are given a conflict-control process

$$\dot{z} = Az + \varphi(u, v), \quad z \in R^n, \quad u \in U, \quad v \in V \tag{1.1}$$

where A is a square matrix of order n, φ: $U \times V \to R^n$ is a function jointly continuous in all its variables, and U and V are non-empty compact sets in Euclidean space R^n.

In addition to the dynamics (1.1), a payoff function $\sigma(z)$, s: $R^n \to R^1$ is given, whose value determines the time at which the game terminates. If $z(t) = z(z_0, u_t(\cdot), v_t(\cdot))$ is the trajectory of system (1.1) corresponding to an initial state z_0 and controls $u_t(\cdot) = \{u(\tau){:}u(\tau) \in U, \tau \in [0, t]\}$, $v_t(\cdot) = \{v(\tau){:}\ v(\tau) \in V, \tau \in [0, t]\}$ chosen by the players, we shall assume that the game terminates at time T if

$$\sigma(z(T)) \leqslant 0 \tag{1.2}$$

The goal of the pursuer (u) is to ensure that the game terminates; that of the evader (v) is the opposite.

We shall assume that the strategies used by the pursuer and evader during the game are Lebesgue-measurable functions of time. Taking the pursuer's part, we shall indicate sufficient conditions which, if satisfied by the parameters of the process (1.1) and the terminal payoff function $\sigma(z)$, guarantee termination of the game (1.1), (1.2). At the same time we shall find the guaranteed termination time, based on information about the initial state z_0 and the prehistory of the control implemented by the evader $v_t(\cdot)$.

We shall assume that the payoff function $\sigma(z)$ is convex and satisfies a Lipschitz condition

$$|\sigma(z) - \sigma(x)| \leqslant 1\|z - x\|, \quad 1 \geqslant 0, \quad z, x \in R^n \tag{1.3}$$

We know from convex analysis [1] that $\sigma(z)$ can be expressed as

$$\sigma(z) = \max_{p \in \operatorname{dom} \sigma^*} [(p, z) - \sigma^*(p)] \tag{1.4}$$

where $\sigma^*(p)$, σ^*: $R^n \to R^1$ is the conjugate of $\sigma(z)$, defined by

$$\sigma^*(p) = \sup_{z \in R^n} [(p, z) - \sigma(z)], \quad p \in R^n \tag{1.5}$$

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and dom σ^* is the effective domain [1] of $\sigma^*(p)$, i.e.

$$\operatorname{dom}\sigma^* = \left\{p \in R^n : \sigma^*(p) < +\infty\right\}.$$

It follows from (1.3) that dom σ^* is a compact set [1].

We require additionally that the payoff function $\sigma(z)$ should be bounded below. Then, in view of (1.5), we have

$$-\sigma^*(p) = \inf_{z \in R^n} \sigma(z)$$

and so dom σ^* contains zero.

Let L be the linear span of the set dom σ^* [1] and let π be the operator of orthogonal projection from R^n onto the subspace L. Using (1.4), we can verify the relation

$$\sigma(z) = \sigma(\pi z), \quad z \in R^n \tag{1.6}$$

2. SCHEME OF THE METHOD AND MAIN RESULT

We introduce the following multivalued mappings

$$W(t,v) = \bigcup_{u \in U} W(t,u,v), \quad W(t) = \bigcap_{v \in V} W(t,v)$$

$$(W(t,u,v) = \pi\Phi(t)\varphi(u,v), \quad \Phi(t) = \exp(tA), t \geqslant 0)$$

Let us assume that the parameters of the process (1.1) satisfy Pontryagin's condition [2, 6], which means that $W(t) \neq \varnothing$ for all $t \geqslant 0$.

Since the mapping $W(t)$ is upper semicontinuous [2], it contains at least one Borelian selector [2, 7, 10]. Denoting the set of all such selectors by Γ, we fix one of them $\gamma(\cdot) \in \Gamma$, and put

$$\xi(t,z,\gamma(\cdot)) = \pi\Phi(t)z + \int_0^t \gamma(t)d\tau$$

We define the resolvent function by

$$\beta(t,\tau,z,v,\gamma(\cdot)) = \sup\Big\{\beta \geqslant 0 : \min_{u \in U} \max_{p \in \operatorname{dom}\sigma^*}\big[(p, W(t-\tau,u,v) - \gamma(t-\tau)) +$$

$$+\beta\big[(p,\xi(t,z,\gamma(\cdot))) - \sigma^*(p)\big]\big] \leqslant 0\Big\} \tag{2.1}$$

$$t \geqslant \tau \geqslant 0, \quad z \in R^n, \quad v \in V$$

It follows at once from Pontryagin's condition that

$$\min_{u \in U} \max_{p \in \operatorname{dom}\sigma^*} (p, W(t-\tau,u,v) - \gamma(t-\tau)) \leqslant 0 \tag{2.2}$$

for all $t \geqslant \tau \geqslant 0, v \in V$.

Consequently, if Pontryagin's condition holds, the inequality in (2.1) is true for at least the zero value of the resolvent function. Note, moreover, that if $\sigma(\xi(t, z, \gamma(\cdot))) \leqslant 0$, then $\beta(t, \tau, z, v, \gamma(\cdot)) = +\infty$ for all $v \in V, \tau \in [0, t]$. But if $\sigma(\xi(t, z, \gamma(\cdot))) > 0$ for some $t > 0, z \in R^n, \gamma(\cdot)) \in \Gamma$, then the resolvent function (2.1) takes finite values and is bounded uniformly in $\tau \in [0, t], v \in V$.

Lemma 2.1. Assume that the parameters of the process (1.1) satisfy Pontryagin's condition and the payoff function $\sigma(z)$ satisfies the conditions of Section 1. Then, if it is true for some $t > 0, z \in R^n$ and $\gamma(\cdot) \in \Gamma$ that $\sigma(\xi(t, z, \gamma(\cdot))) > 0$, the function

$$\beta(\tau,v) = \beta(t,\tau,z,v,\gamma(\cdot)) \tag{2.3}$$

is Borelian jointly with respect to the variables (τ, v) on the set $[0, t] \times V$.

Proof. Fix values $t > 0, z \in R^n$ and $\gamma(\cdot) \in \Gamma$ for which $\sigma(\xi(t, z, \gamma(\cdot))) > 0$. It follows from our assumptions about the parameters of the process (1.1) and the payoff function $\sigma(z)$ that the function $\psi(u, v, p, \tau, \gamma, \beta) = (p, W(t-\tau,$

$u, v) - \gamma) + \beta[(p, \xi(t, z, \gamma(\cdot))) - \sigma^*(p)]$ is jointly continuous in its variables, so that [8, 11] the same is true of the function

$$\Psi(\nu,\tau,\gamma,\beta) = \min_{u\in U} \max_{p\in \operatorname{dom}\sigma^*} \psi(u,\nu,p,\tau,\gamma,\beta)$$

Then, by [11], the multivalued mapping

$$B(\tau,\nu,\gamma) = \{\beta \geq 0 : \Psi(\tau,\nu,\gamma,\beta) \leq 0\}$$

is upper continuous and its selector

$$\beta(\tau,\nu,\gamma) = \sup\{\beta : \beta \in B(\tau,\nu,\gamma)\}$$

is Borelian jointly in its variables. Therefore, by a property of a superposition of two Borelian functions [10], the function (2.3) is Borelian on the set $[0, t] \times V$.

Consider the function

$$T(z,\gamma(\cdot)) = \inf\left\{ t \geq 0 : \int_0^t \inf_{\nu\in V} \beta(t,\tau,z,\nu,\gamma(\cdot))d\tau \right. \tag{2.4}$$

If the inequality in braces in (2.4) does not hold for any $t \geq 0$, we set $T(z, \gamma(\cdot)) = +\infty$. Note that if $\sigma(\xi(t, z, \gamma(\cdot))) > 0$, then

$$\inf_{\nu\in V} \beta(t,\tau,z,\nu,\gamma(\cdot))$$

is a measurable function of τ, $\tau \in [0, t]$ [2], and since it is bounded uniformly in τ, it is also summable over the interval $[0, t]$. But if $\sigma(\xi(t, z, \gamma(\cdot))) < 0$, the integrand in (2.4) is equal to $+\infty$ for all $\tau \in [0, t]$, $t > 0$. Therefore it is natural to define the integral as $+\infty$, and so the inequality in the definition of the function $T(z, \gamma(\cdot))$ is automatically satisfied.

Theorem 2.1. Assume that the parameters of the process (1.1) satisfy Pontryagin's condition and the payoff function $\sigma(z)$ satisfies the conditions of Section 1. Assume that for some $z^0 \in R^n$, $\gamma^0(\cdot) \in \Gamma$ we have $T(z^0, \gamma^0(\cdot)) < +\infty$. Then the game may terminate at time $T(z^0, \gamma^0(\cdot))$ from the initial position z^0.

Proof. Set $T = T(z^0, \gamma^0(\cdot))$. Let $\nu(\tau)$, $\nu(\tau) \in V$, $\tau \in [0, T]$ be an arbitrary measurable function. We shall show how to choose the pursuer's control.

Let us consider the case $\sigma(\xi(T(z^0, \gamma^0(\cdot))) > 0$. Define the control function by

$$h(t) = 1 - \int_0^t \beta(T,\tau,z^0,\nu(\tau),\gamma^0(\cdot))d\tau \tag{2.5}$$

The function $h(t)$ is continuous, non-increasing and $h(0) = 1$. It follows from the definition of T that $t_* = t_*(\nu(\cdot))$, $0 < t_* < T$ exists, such that $h(t_*) = 0$.

Consider the multivalued mapping

$$U_1(\tau,\nu) = \left\{ u \in U : \max_{p\in\operatorname{dom}\sigma^*} \left[\left(p, W(T-\tau,u,\nu) - \gamma^0(T-\tau)\right) + \right.\right.$$
$$\left.\left. +\beta(T,\tau,z^0,\nu,\gamma^0(\cdot))[(p,\xi(T,z^0,\gamma^0(\cdot))) - \sigma^*(p)]\right] \leq 0 \right\}$$
$$0 \leq \tau \leq t_*, \nu \in V$$

It is Borelian jointly in all its variables.

Indeed, the function

$$\kappa(u,\nu,\tau,\gamma,\beta) = \max_{p\in\operatorname{dom}\sigma^*} \left[(p, W(T-\tau,u,\nu) - \gamma) + \beta[(p,\xi(T,z^0,\gamma^0(\cdot))) - \sigma^*(p)] \right]$$

is jointly continuous in its variables. Therefore, the mapping

$$U(\tau, v, \gamma, \beta) = \{u \in U : \kappa(u, v, \tau, \gamma, \beta) \leqslant 0\}$$

is upper semicontinuous [11], and so the multivalued mapping

$$U_1(\tau, v) = U(\tau, v, \gamma(T-\tau), \beta(T, \tau, z^0, v, \gamma^0(\cdot)))$$

is Borelian jointly in (τ, v) as the superposition of a semicontinuous function and a Borelian mapping [10].

Then the selector

$$u_1(\tau, v) = \text{lex min}\, U_1(\tau, v), \quad 0 \leqslant \tau \leqslant t_*, \quad v \in V$$

is a Borelian function of (τ, v) [2, 12].

Consider the multivalued mapping

$$U_2(\tau, v) = \{u \in U : W(T-\tau, u, v) - \gamma^0(T-\tau) = 0\}, \quad t_* \le \tau \leqslant T, \quad v \in V$$

It is jointly Borelian in (τ, v) [2, 7]. Then the selector

$$u_2(\tau, v) = \text{lex min}\, U_2(\tau, v), \quad t_* \le \tau \leqslant T, \quad v \in V$$

is a Borelian function of (τ, v) [2, 12].

Let us define the pursuer's control in the interval $[0, T]$ as

$$u(\tau) = \begin{cases} u_1(\tau, v(\tau)), & \tau \in [0, t_*) \\ u_2(\tau, v(\tau)), & \tau \in [t_*, T] \end{cases}$$

The function $u(\tau)$ is measurable [2, 10].

Now let us consider the case $\sigma(\xi(T, z^0, \gamma^0(\cdot))) \leqslant 0$. The pursuer's control over $[0, T]$ is defined as

$$u(\tau) = u_2(\tau, v(\tau))$$

This function is also measurable [2, 10].

If the pursuer chooses the control law in this way, then, whatever controls the evader adopts at time T, inequality (1.2) will be true along the relevant trajectories of the process (1.1).

Indeed, Cauchy's formula for the process (1.1) implies a representation

$$\pi z(T) = \xi(T, z^0, \gamma^0(\cdot)) + \int_0^T \left[W(T-\tau, u(\tau), v(\tau)) - \gamma^0(T-\tau)\right] d\tau \tag{2.6}$$

Let $\sigma(\xi(T, z^0, \gamma^0(\cdot))) < 0$. Because of the pursuer's control law, we have

$$\pi z(T) = \xi(T, z^0, \gamma^0(\cdot))$$

Hence the truth of (1.2) follows at once from (1.6).

Now let $\sigma(\xi(T(z^0, \gamma^0(\cdot))) > 0$. It follows from (1.6) and (2.6) that

$$\sigma(z(T)) = \max_{p \in \text{dom}\,\sigma^*} \left[(p, \xi(T, z^0, \gamma^0(\cdot))) - \sigma^*(p) + \int_0^T (p, W(T-\tau, u(\tau), v(\tau)) - \gamma^0(T-\tau)) d\tau\right]$$

Adding and subtracting the expression

$$\left[(p, \xi(T, z^0, \gamma^0(\cdot))) - \sigma^*(p)\right](I - h(t_*))$$

inside the square brackets, we obtain a relationship from which it follows that the pursuer, by choosing the control as indicated, may guarantee that at time T

$$\sigma(z(T)) \leqslant \sigma(\xi(T,\ z^0,\ \gamma^0(\cdot)))h(t_*)=0$$

Corollary 2.1. Suppose that the parameters of the process (1.1) satisfy Pontryagin's condition and the payoff function $\sigma(z)$ satisfies the conditions of Section 1. Then, if the pursuer uses the control laws described in the proof of the theorem, the following estimate will hold for any T, $0 < T < T(z^0, \gamma^0(\cdot))$

$$\sup_{v(\cdot)\in\Omega_V} \sigma(z(T)) \leqslant \sigma(\xi(T,z^0,\gamma^0(\cdot)))\left[1-\int_0^T \inf_{v\in V} \beta(T,\tau,z^0,v,\gamma^0(\cdot))d\tau\right] \tag{2.7}$$

where Ω_V is the set of all measurable functions with values in V.

The proof is analogous to that of the theorem, except that the control function must be defined as the difference of the function $h(t)$ defined above in (2.5) and the quantity in square brackets in (2.7).

3. GENERALIZED DISTANCE

Let M^* be a convex set and let S be a convex bounded set whose interior contains zero. Then for all $z \in R^n$ one can define a generalized distance function as [8]

$$d_S(z|M^*) = \inf\left\{p \geqslant 0 : z \in M^* + \rho S\right\} = \sigma(z)$$

It can be shown that this function satisfies the conditions of Section 1.

Let us calculate the conjugate $\sigma^*(p), p \in R^n$, of the generalized distance function $d_s(z|M^*)$.

We first note that

$$d_S(z|M^*) = \inf\left\{\mu_S(z-m) : m \in M^*\right\}$$

where $\mu_s(x) = \inf\{\mu > 0 : x \in \mu S\}$ is the gauge of S [1, 8]. Therefore, starting from the definition of the infimal convolution operation [1, 7], we have

$$d_S(z|M^*) = (f \,\Box\, g)(z) = \inf\left\{f(z-y)+g(y) : yR^n\right\}$$

where $\Box$ denotes infimal convolution [1], $f(x) = \mu_S(x)$ is the gauge of $S, x \in R^n$ and $g(y) = \delta(y\,|M^*)$ is the indicator of the set M^* [1].

By the duality theorem for addition and infimal convolution [1], we obtain a formula for the conjugate function

$$\sigma^*(p) = d_S^*(|M^*)(p) = f^*(p)+g^*(p) = \begin{cases} C(M^*,p), & p \in S^0 \\ +\infty, & p \notin S^0 \end{cases}$$

where $S^0 = \{p \in R^n : (p,x) \leqslant 1 \text{ for every } x \in S\}$ is the polar of the set S [1], $f^*(p) = \delta(p\,|S^0)$ is the indicator of the polar of S and $g^*(p) = C(M^*, p)$ is the support function of M^*.

We have used the fact that the gauge of S is the support function of the polar S^0 [1], as well as the duality of the indicator and support function of a convex closed set [1].

Thus, taking (1.4) into account, we have

$$\operatorname{dom} d_S^*(|M^*)(p) = S^0$$

$$d_S(z|M^*) = \max_{p\in S^0}\left[(p,z) - C(M^*,p)\right]$$

Using this representation, one can readily prove the following lemma.

Lemma 3.1. Let X be a compact set, let M^* be a convex closed set, and let S be a convex bounded set whose interior contains zero. Then the necessary and sufficient condition for the truth of $X \cap M^* \neq \phi$ is

$$\min_{z\in X}\max_{p\in S^0}\left[(p,z) - C(M^*,p)\right] \leqslant 0$$

where S^0 is the polar of S.

Take M^* to be a cylindrical set of the form $M^* = M_0 + M$, where M_0 is a linear subspace of R^n and M is a convex compact subset of the orthogonal complement L of M_0 in R^n.

Then the defining formula (2.1) yields an expression for the resolvent function $\beta(t, \tau, z, v, \gamma(\cdot))$, namely

$$\sup\Big\{\beta \geqslant 0: \min_{u \in U} \max_{p \in S^0 \cap L} [(p, W(t-\tau, u, v) - \gamma(t-\tau)) +$$

$$+\beta[(p, \xi(t, z, \gamma(\cdot))) - C(M, p)]] \leqslant 0\Big\}, \quad t \geqslant \tau \geqslant 0, \quad z \in R^n, v \in V$$

where S is a convex compact set in R^n whose interior contains the zero of the space.

Using Lemma 3.1, it can be shown that this function is precisely the resolvent function $\alpha(t, \tau, z, v, \gamma(\cdot))$ defined in [2] by

$$\sup\{\alpha \geqslant 0: [W(t-\tau, v) - \gamma(t-\tau)] \cap \alpha[M - \xi(t, z, \gamma(\cdot))] \neq \phi\}$$

We have thus established the connection between the results obtained here and the general scheme of the resolvent-function method.

REFERENCES

1. ROCKAFELLAR R. T., *Convex Analysis*. Princeton University Press, Princeton, NJ, 1970.
2. CHIKRII A. A., *Conflict-control Processes*. Naukova Dumka, Kiev, 1992.
3. CHIKRII A. A., Minkowski functionals in pursuit theory. *Dokl. Akad. Nauk Rossii* **329,** 3, 281–284, 1993.
4. CHIKRII A. A. and RAPPOPORT I. S., Gauge functions in differential games. *Avtomatika* 6, 9–16, 1992.
5. KRASOVSKII N. N. and SUBBOTIN A. I., *Positional Differential Games*. Nauka, Moscow, 1974.
6. PONTRYAGIN L. S., *Selected Scientific Papers*, Vol. 2, *Differential Equations*. Nauka, Moscow, 1974.
7. IOFFE A. D. and TIKHOMIROV V. M., *Theory of Extremal Problems*. Nauka, Moscow, 1974.
8. PSHENICHNYI B. N., *Convex Analysis and Extremal Problems*. Nauka, Moscow, 1980.
9. AUBIN J.-P. and EKELAND I., *Applied Nonlinear Analysis*. John Wiley, New York, 1984.
10. KURATOWSKI K., *Topology*, Vol. 1. PWN, Warsaw [and Academic Press, New York], 1966.
11. FEDOROV V. V., *Numerical Maximin Methods*. Nauka, Moscow, 1979.
12. WARGA J., *Optimal Control of Differential and Functional Equations*. Academic Press, New York, 1972.

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